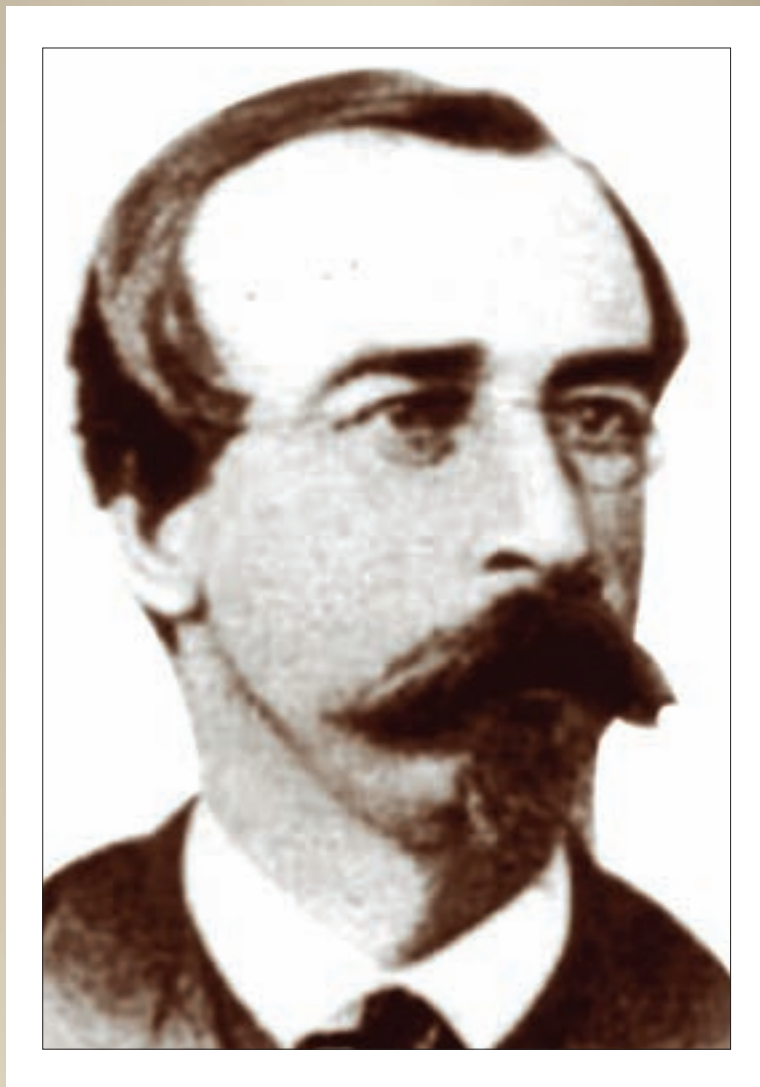


Laguerre Mathematics in Optical Communications

Mario Martinelli and Paolo Martelli



The mathematical studies done by Edmond Nicolas Laguerre in the 19th century laid the foundation for contemporary optical communications.

Laguerre's name comes up in the world of optical communications as many times as Fourier's does in diffraction optics. Perhaps one of the first times his name was invoked was in 1965, when the laser was in its infancy. At an important Congress of that pioneer period, Roy J. Glauber noticed that the photocount distribution of a monomodal field, which results from the superposition of a coherent excitation and a chaotic one, presented a generating function with "the same form as the generating function for the Laguerre polynomials." (The conference, titled "The Physics of Quantum Electronics," took place in San Juan, Puerto Rico, on June 28-30, 1965.) As a consequence, he derived a photocount distribution expressed in terms of the Laguerre polynomials $L_n(x)$, which are defined by:

$$L_n(x) = \sum_{m=0}^n (-1)^m \binom{n}{n-m} \frac{x^m}{m!} \quad (1)$$

At the same conference, M. Scully, W.E. Lamb and M.J. Stephen noticed that laser light produced photon statistics that were quite different than the case of blackbody radiation; in fact, they approached Poisson statistics.

In those years, the photon statistics had been deeply investigated, both theoretically and experimentally. Glauber had just proposed his model of "coherent states," for which he was awarded the Nobel Prize in Physics in 2005, and experimentalists subsequently confirmed his theoretical predictions.

Again in 1965, Gerard Lachs, in a seminal paper published in the May *Physical Review* (and hence before the San Juan Conference) treated the "theoretical aspects of mixtures of ther-

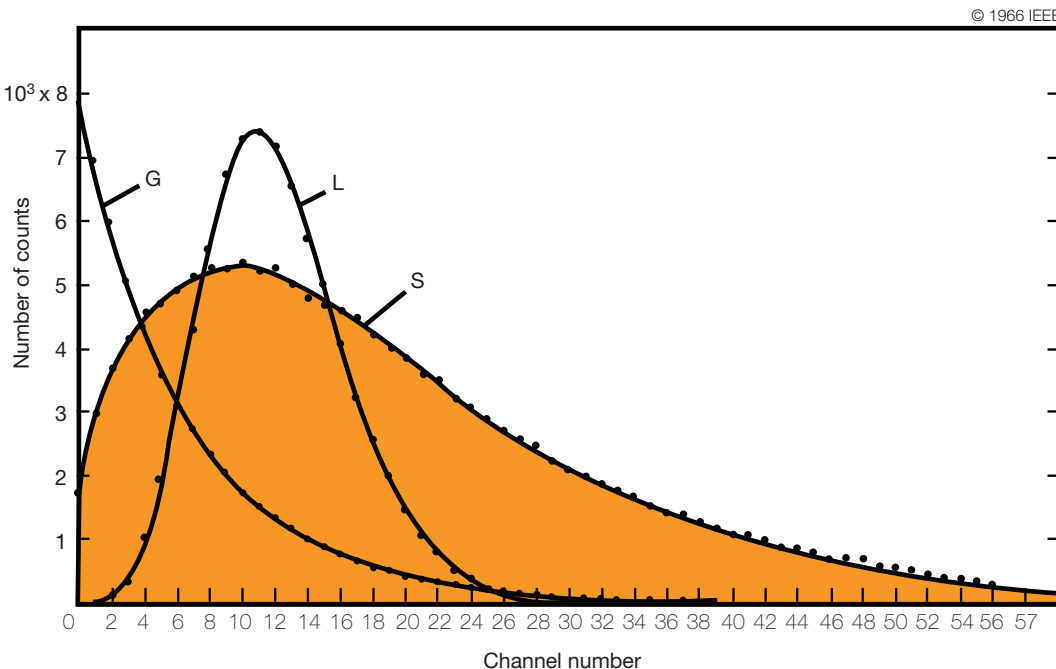
[Laguerre polynomials up to the 4 th order]
$L_0(x) = 1$
$L_1(x) = 1 - x$
$L_2(x) = 1 - 2x + \frac{1}{2}x^2$
$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$
$L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4$

mal and coherent radiation." He proposed that the photocount distribution $p(n)$ should have an analytical expression, in which the confluent hypergeometric function ${}_1F_1(-n, 1, x)$ appeared.

Indeed, the confluent hypergeometric function, with its first argument negative and equal to $-n$ and second argument equal to unity, can be rewritten as a Laguerre polynomial $L_n(x)$ of order n , as also shown in the fundamental book by R.M. Gagliardi and S. Karp a few years later. So the expression of the photocount probability for "mixed light" is the following Laguerre distribution:

$$p(n) = \frac{\mu_{th}^n}{(1 + \mu_{th})^{n+1}} \cdot \exp\left(-\frac{\mu_{cob}}{1 + \mu_{th}}\right) \cdot L_n\left[-\frac{\mu_{cob}}{\mu_{th}(1 + \mu_{th})}\right] \quad (2)$$

[The experimental photon counting distribution]



The experimental results reported in this figure (© 1966 IEEE), from F.T. Arecchi et al. IEEE J. Quantum Electron. **2**, 341-50, confirmed the theoretical predictions of R.J. Glauber: The curve G is the typical Bose-Einstein photon distribution of a "thermal" source (chaotic light); the curve L is the predicted Poisson distribution of the laser source well above threshold (coherent light) and the curve S is the curve pertinent to a mixture between chaotic and coherent light. All the curves are well represented by an expression containing the Laguerre polynomials (see equation 2 in the text).

The statistical properties of the superposition of coherent and chaotic light became very relevant to optical communications when the optical amplifier appeared as a key element in fiber-optic systems.

where μ_{cob} and μ_{th} are the average photocounts of the coherent and thermal (or more in general “chaotic”) part of the mixed light, respectively. An interesting aspect of the equation, as shown in the figure below, is that, when the strength of the coherent light goes to zero, the distribution reduces the Bose-Einstein distribution; however, when the chaotic light decreases, the distribution converges to the Poisson distribution of the coherent light.

Actually, equation (2) refers to the mixture of monomodal chaotic light and coherent light. The monomodality condition is fulfilled for a polarized component of the radiation emitted by a spatially coherent thermal source, when the length of time T of the observation interval in the photon counting measurement—i.e., the observation time—is much less than the coherence time τ_c of the source.

The photocount probability can be generalized to the case of superposition of multimodal chaotic light and coherent light by means of the associated Laguerre polynomials, defined as

$$L_n^{(k)}(x) = \sum_{m=0}^n (-1)^m \binom{n+k}{n-m} \frac{x^m}{m!}, \quad (3)$$

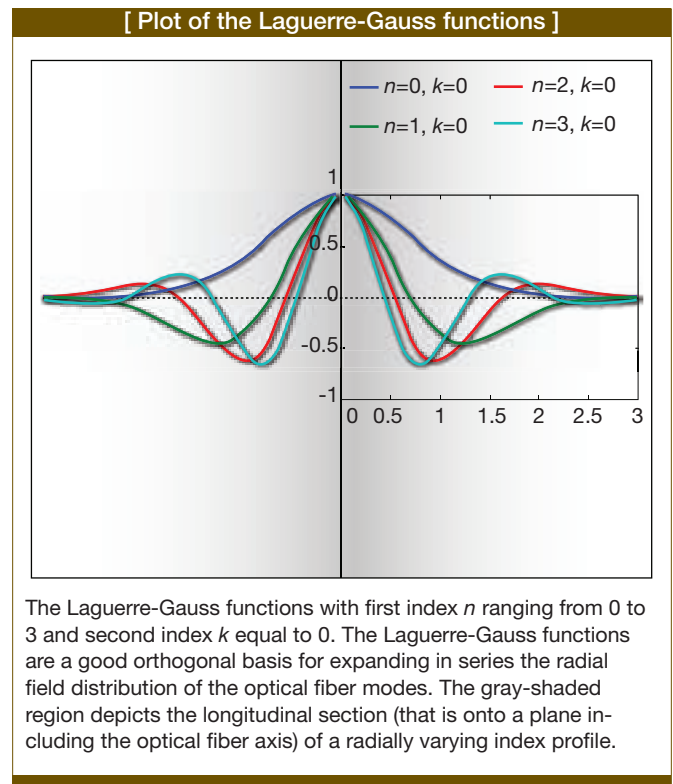
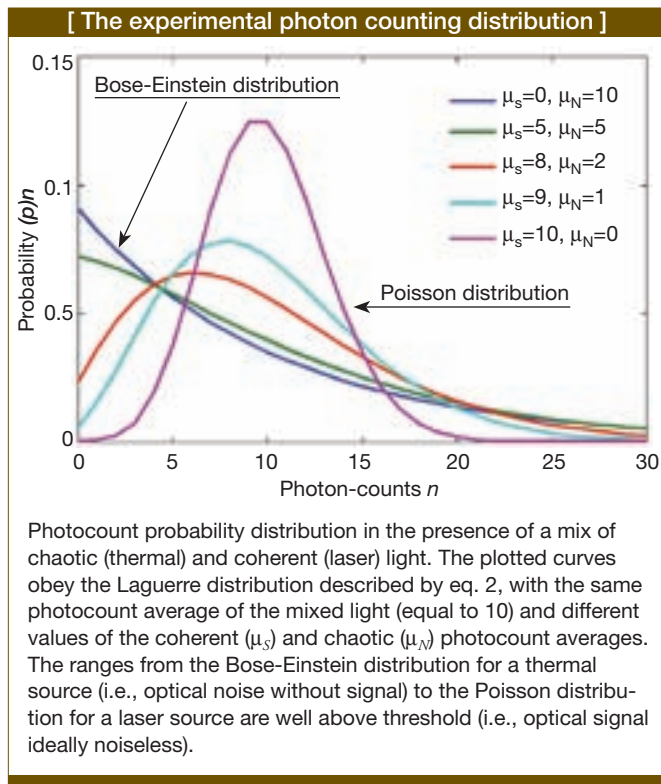
and which can be easily calculated by the recurrence relation

$$(n+1) \cdot L_{n+1}^{(k)}(x) + (x - k - 2n - 1) \cdot L_n^{(k)}(x) + (n+k) \cdot L_{n-1}^{(k)}(x) = 0. \quad (4)$$

The Laguerre distribution becomes, in the general case of the coherent and M -mode chaotic light mixture:

$$p(n) = \left(\frac{\mu_{th}}{M}\right)^n \cdot \left(1 + \frac{\mu_{th}}{M}\right)^{-n-M} \cdot \exp\left(-\frac{M \mu_{cob}}{M + \mu_{th}}\right) \cdot L_n^{(M-1)}\left[\frac{M^2 \mu_{cob}}{\mu_{th}(M + \mu_{th})}\right]. \quad (5)$$

For spatially coherent polarized thermal light, the number of modes M can be approximated by the ratio T/τ_c , when the observation time is much greater than the coherence time for a high number of modes. In this case, one can intuitively picture T/τ_c as the number of statistically independent samples of the optical intensity in an observation interval and identify



this ratio as the number of temporal modes. A more precise expression for the number of temporal modes, and hence for M , in the case of a single polarization mode and single spatial mode, is

$$M = \frac{T^2}{2} \left[\int_0^T (T-\tau) |\gamma(\tau)|^2 d\tau \right]^{-1}, \quad (6)$$

where $\gamma(\tau)$ is the normalized autocorrelation function of the optical field amplitude (i.e., the degree of first-order coherence) of the chaotic component.

The statistical properties of the superposition of coherent and chaotic light became very relevant to optical communications when the optical amplifier appeared as a key element in fiber-optic systems. This is because “mixed light” is exactly the light obtained after the optical amplification of coherent light. In fact, the optical amplifier increases the coherent signal by stimulated emission of photons. However, it also introduces an unavoidable amount of “chaotic” light given by amplified spontaneous emission (ASE).

Hence, the Laguerre distribution provides a synthetic but powerful law that describes the photon statistical properties of amplified optical communication signals. In particular, from the photon number average and variance, the degradation of the optical signal-to-noise ratio (OSNR) can be evaluated, and the noise figure of the optical amplifier determined.

For a long time, the Laguerre polynomials remained hidden to everyone but specialists, until they were recognized by E. Schrödinger in 1926 as a solution for the radial part of “his” equation when applied to the hydrogen atoms. By separating the angular and radial variables, Schrödinger solved a second-order differential equation in the function $R(r)$, where r represents the distance of the electron from the nucleus.

Schrödinger discovered that this equation admits as eigensolutions the *Laguerreschen orthogonalfunktionen*—that is, the orthogonal functions obtained from the associated Laguerre polynomials. In fact, mathematicians know that, by using Laguerre polynomials, it is possible to recover useful orthogonal functions called associated Laguerre functions

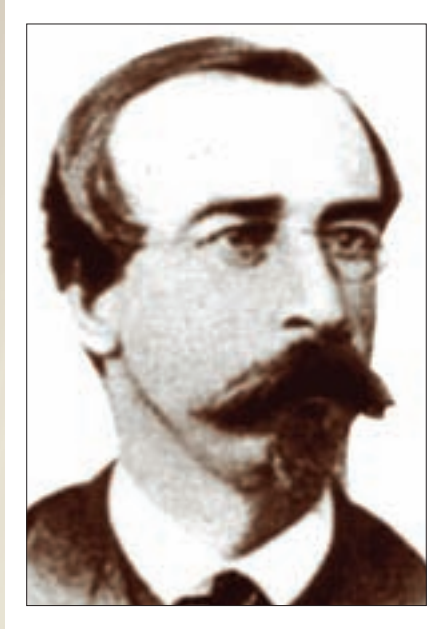
$$\Psi_n^{(k)}(x) = e^{-x/2} x^{k/2} L_n^{(k)}(x). \quad (7)$$

These satisfy the following orthogonality condition

$$\int_0^\infty e^{-x} x^k L_n^{(k)}(x) L_m^{(k)}(x) dx = \frac{(n+k)!}{n!} \delta_{m,n}. \quad (8)$$

Hence, Schrödinger found another very important use of the Laguerre polynomials in the associated orthogonal functions, which are the eigenfunction of the quantum structure (the hydrogen atom) described by the differential equation under examination.

Surprisingly, the associated Laguerre functions have been re-discovered as a solution to a problem that is completely different from the one analyzed by Schrödinger from a physical point of view, and yet very similar to it mathematically: the problem



Who was Laguerre?

Edmond Nicolas Laguerre was born in 1834 in a small French village called Bar-Le-Duc. In 1853, he applied to the Ecole Polytechnique, the already famous military French college. After various vicissitudes, he remained there as an assistant and then as a “répétiteur” of the course of mathematical analysis. In 1885, the year before he died, he was elected to the French Académie des Science with the sponsorship of Camille Jordan.

During his relatively short life, he published a remarkable 140 papers on the principal branches of mathematics. One of these papers, which was published in 1879 in *Bulletin de la Société Mathématique de France*, introduced what are now called the Laguerre polynomials.

Laguerre was mainly interested in projective geometry, differential geometry and algebra methods. He didn’t spend much time considering the polynomials that made him so famous.

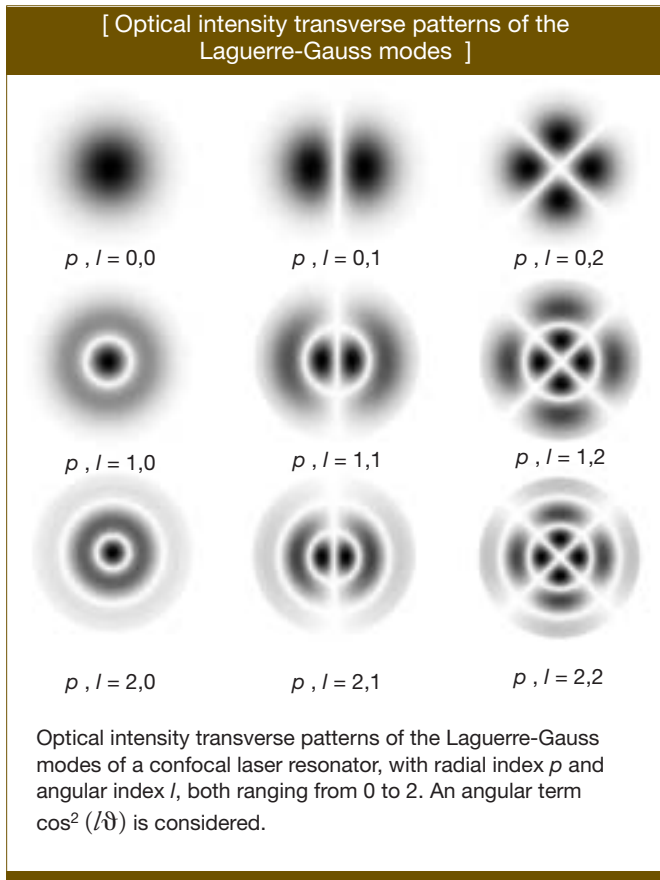
The Laguerre-Gauss functions can be a very suitable basis for expanding in series the radial field distribution of the guided modes propagating in optical fiber, regardless of the refractive-index profile.

of the propagation of light beams in a gradient-index-guiding media.

In 1967, W. Streifer and C.N. Kurtz recognized that the scalar wave equation, applied to a radially inhomogeneous media with a parabolic profile of the square of the refractive index, admits as exact solutions (corresponding to the guided modes) the Laguerre-Gauss functions $\varphi_{nk}(\rho)$; these are just the associated Laguerre functions with the square of the distance ρ from the propagation axis as argument, apart from a multiplicative coefficient:

$$\varphi_{nk}(\rho) = \left(\frac{\rho}{\rho_0}\right)^k \cdot e^{-\frac{1}{2}\left(\frac{\rho}{\rho_0}\right)^2} \cdot L_n^{(k)}\left(\frac{\rho^2}{\rho_0^2}\right). \quad (9)$$

For example, the Laguerre-Gauss functions with first index n ranging from 0 to 3 and with second index k equal to 0 are



plotted in the figure on the bottom right of p. 32, as a function of the normalized radial distance.

In the pioneer years of fiber optics, the Laguerre-Gauss functions were recognized as an orthogonal basis for approximating the modes of weakly guiding fibers with a radially varying index of refraction—an alternative to the orthogonal Bessel functions, which are solutions for step-index fibers.

Contemporary fiber optics typically operate with small refractive index variation in the weakly guiding approximation, and with index profiles of complicated shapes that are very different from the simple step-index profile. Therefore, it seems that the Laguerre-Gauss functions can be a very suitable basis for expanding in series the radial field distribution of the guided modes propagating in optical fiber, regardless of the refractive-index profile.

But fiber optics is not the first area within optics to draw on Laguerre mathematics. The Laguerre-Gauss functions have already played a role in the contiguous field of the propagation of laser beams in free space. In fact, in the early years of the laser, scientists paid much attention to describing the exact shapes of the light beam emerging from laser resonators.

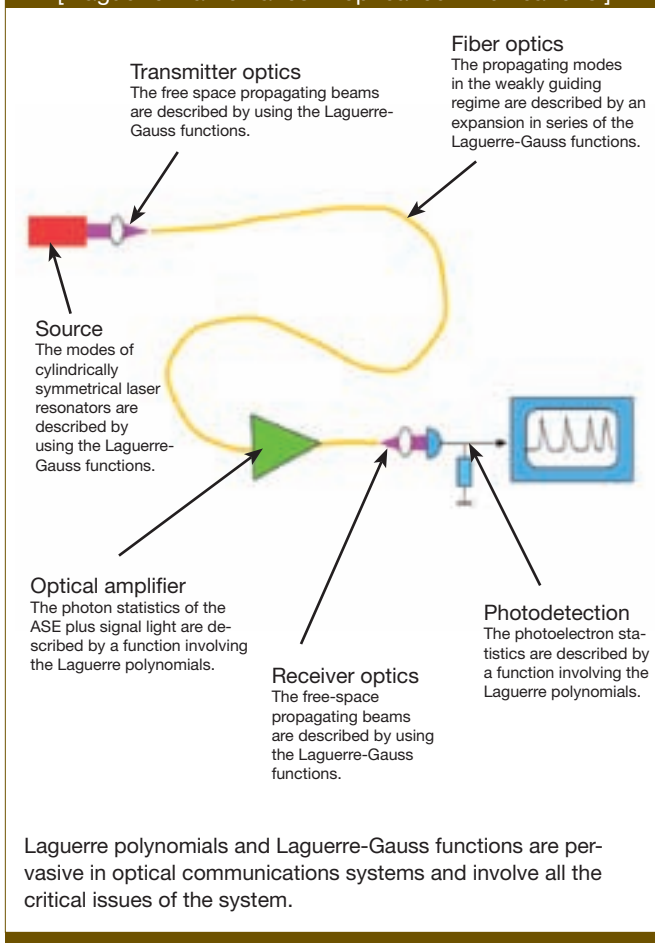
In this context, G.D. Boyd and H. Kogelnik pointed out in 1962 that the radial field distributions of the modes of a confocal resonator in cylindrical coordinates, under certain conditions, are given by the eigensolutions of an integral equation with the same kernel of the Hankel transform.

This integral equation can be obtained by using the scalar formulation of Huygens' principle in determining the field propagating back and forth between the reflectors, under the assumptions of large reflectors and field-concentrated near the axis (i.e., paraxial approximation), and by imposing self-reproducing field patterns, for the modes.

They found the field modes on the central transversal plane of the resonator to be the Laguerre-Gauss functions, as these functions are self-reciprocal under the Hankel transformation. The transverse patterns, in terms of optical intensity, of the Laguerre-Gauss mode with first index p (i.e., the radial index) and second index l (i.e., the angular index) both range from 0 to 2, are shown in the figure left in which an angular term $\cos^2(l\vartheta)$ is considered.

Then, the field distribution of the modes inside and outside the confocal resonator can be derived from the Laguerre-Gauss functions by using the Huygens' principle under paraxial

[Laguerre mathematics in optical communications]



approximation—that is, using the Fresnel diffraction integral to obtain the so-called Laguerre-Gauss beams. These beams can also be seen as cylindrically-symmetrical solutions to the paraxial scalar wave equation.

Indeed, G. Gobau and F. Schwering arrived at the same conclusion one year before (1961) while looking for an electromagnetic wave beam in which the cross-sectional amplitude distribution repeats itself at a certain distance after passing through a succession of uniformly spaced phase transformers. In the optical domain, such phase transformers are nothing but common lenses.

In conclusion, if we consider the modern optical communication system as a whole, it is surprising how pervasively the Laguerre mathematics enter into all the vital aspects of the system.

For example, at the transmitter module, a precise description of the laser modes in cylindrically symmetrical resonators (as in certain solid-state lasers or VCSELs) and of the beam laser propagation in cylindrical geometry is given using the Laguerre-Gauss functions.

At the optical fiber, the wave-guided propagation modes can be approximated with high accuracy by expanding in series the

Laguerre-Gauss functions, even in presence of modern refractive-index profiles.

At the receiver, the Laguerre polynomials give a synthetic but powerful and accurate description of the photocount statistics in the presence of chaotic optical noise, providing consequently the mathematical basis for the OSNR evaluation; these are the same statistics that rule the photon distribution produced by an optical amplifier when the ASE noise photons are mixed with the signal-coherent photons.

Hence, after the debt paid by optics to Jean Baptiste Joseph Fourier (another great French mathematician who died four years before Laguerre's birth), it seems that the time has come for optical communications to honor Edmond Nicolas Laguerre. ▲



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[References and Resources]

- >> E. Schrödinger. "Quantisierung als eigenwertproblem - III," *Ann. der Phys.* **80**, 437-90 (1926).
- >> G. Gobau and F. Schwering. "On the guided propagation of electromagnetic wave beams," *IRE Trans. on Antennas and Propagation* **9**, 248-56 (1961).
- >> G.D. Boyd and H. Kogelnik. "Generalized confocal resonator theory," *Bell. Sys. Tech. J.* **41**, 1347-69 (1962).
- >> G. Lachs. "Theoretical aspects of mixtures of thermal and coherent radiation," *Phys. Rev.* **138**, B1012-1016 (1965).
- >> F.T. Arecchi et al. "Photocount distribution and field statistics," *IEEE J. Quantum Electron.* **2**, 341-50 (1966).
- >> R.J. Glauber. "Photon counting and field correlations," in *Proceedings of the Conference on the Physics of Quantum Electronics, Puerto Rico, 1965*, P.L. Kelley, B. Lax, and P.E. Tannenwald, eds. McGraw-Hill Book Co., Inc., N.Y., 788-811 (1966).
- >> M. Scully et al. "Quantum theory of an optical maser," in *Proceedings of the Conference on the Physics of Quantum Electronics, Puerto Rico, 1965*, P.L. Kelley, B. Lax, and P.E. Tannenwald, eds., McGraw-Hill Book Co., Inc., N.Y., 759-67 (1966).
- >> W. Streifer and C.N. Kurtz. "Scalar analysis of radially inhomogeneous guiding media," *J. Opt. Soc. Am.* **57**, 779-86 (1967).
- >> R.M. Gagliardi and S. Karp. *Optical communications*, Wiley-Interscience, New York (1976).
- >> O. Georg. "Use of the orthogonal system of Laguerre-Gaussian functions in the theory of circularly symmetric optical waveguides," *Appl. Opt.* **21**, 141-6 (1982).
- >> C. Brezinski. "Edmond Nicolas Laguerre" in *Polynômes Orthogonaux et Applications, Proceedings, Bar-le-Duc 1984*, C. Brezinski, A. Daux, A.P. Magnus, P. Maroni, and A. Ronveaux, eds., *Lectures Note on Mathematics* **1171**, Springer-Verlag, Berlin, XI-XXVI (1985).